TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 317, Number 1, January 1990

EXTENDING H^p FUNCTIONS FROM SUBVARIETIES TO REAL ELLIPSOIDS

KENZŌ ADACHI

ABSTRACT. Let Ω be a domain in C^n which is a somewhat generalized type of the real ellipsoid. Let V be a subvariety in Ω which intersects $\partial\Omega$ transversally. Then there exists an operator $E: H^p(V) \to H^p(\Omega)$ satisfying $Ef|_V = f$.

1. INTRODUCTION

The H^p extension problem from subvarieties in a strictly pseudoconvex domain was studied by Henkin [10], Adachi [1, 2, 3], and Hatziafratis [9] when $p = \infty$, and then, by Cumenge [6], and Beatrous [4] when $p \in (0, \infty)$. Let D be a real ellipsoid, i.e.,

$$D = \left\{ x + iy \in C^N : \sum_{i=1}^{N} x_i^{2n_i} + \sum_{i=1}^{N} y_i^{2m_i} < 1 \right\},$$

where $n_1, \dots, n_N$ and $m_1, \dots, m_N$ are positive integers. Then Diederich-Fornaess-Wiegerinck [7] proved Hölder estimates for solutions of $\bar{\partial}$ problem in D. In their proof, they used the explicit integral formula which involves the support function $\Phi(\zeta, z)$, depending holomorphically on z. On the other hand, Hatziafratis [8] constructed the integral formula for holomorphic functions in a subvariety of a bounded pseudoconvex domain with smooth boundary. His results are the extension of an earlier work of Stout [12]. In the present paper, by applying the integral formula constructed by Hatziafratis, we study the H^p extension from subvarieties in a convex domain Ω, which is a somewhat generalized type of the real ellipsoid D. Finally, we shall adopt the convention of denoting by c any positive constant which does not depend on the relevant parameters in the estimate.

2. PRELIMINARIES AND RESULTS

Let $s_i(x_i)$, $t_i(y_i)$ be real analytic functions on $[0, a]$. We set

$$\phi_i(x_i) = s_i(x_i^2) \quad \text{and} \quad \psi_i(y_i) = t_i(y_i^2).$$

Received by the editors November 1, 1987 and, in revised form, June 21, 1988.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 32A25, 32A35.

©1990 American Mathematical Society
0002-9947/90 $1.00 + $.25 per page

Suppose that ϕ_i, ψ_i $(i = 1, \dots, N)$ satisfy the following conditions:

(i) $\phi_i''(x_i) \geq 0$, $\psi_i''(y_i) \geq 0$ for $i = 1, \dots, N$.

(ii) $\phi_i(a) > 1$, $\psi_i(a) > 1$, $\phi_i(0) = \psi_i(0) = 0$ for $i = 1, \dots, N$.

(iii) $\phi_i''(x_i) > 0$ for $x_i \neq 0$, $\psi_i''(y_i) > 0$ for $y_i \neq 0$.

We set

$$\rho(z) = \sum_{i=1}^{N} \phi_i(x_i) + \sum_{i=1}^{N} \psi_i(y_i) - 1 \quad \text{for } z = (x_1 + iy_1, \dots, x_N + iy_N).$$

Let

$$\Omega = \{z \in C^N : \rho(z) < 0\}.$$

Let $\widetilde{V}$ be a subvariety in a neighborhood $\widetilde{\Omega}$ of $\overline{\Omega}$ which intersects $\partial\Omega$ transversally. Suppose that $\widetilde{V}$ is written in the following form:

$$\widetilde{V} = \{z \in \widetilde{\Omega} : h_1(z) = \cdots = h_m(z) = 0\} \qquad (m < N),$$

where $h_1, \dots, h_m$ are holomorphic functions in $\widetilde{\Omega}$ which satisfy

$$\partial h_1 \wedge \cdots \wedge \partial h_m \wedge \partial \rho \neq 0 \quad \text{on } \widetilde{V} \cap \partial\Omega.$$

Let $V = \widetilde{V} \cap \Omega$. For any bounded domain G with smooth boundary in a complex variety, we denote by $H^p(G)$ the Hardy class on G. Then we have

Theorem 1. *Let $f \in H^p(V)$ $(1 \leq p < \infty)$. Then there exists a function $F \in H^p(\Omega)$ such that $F(z) = f(z)$ for $z \in V$.*

Theorem 2. *Suppose that V has no singular points. Let f be a holomorphic function on V with $\int_V |f|^p \, d\sigma < \infty$ $(1 \leq p < \infty)$. Then there exists a holomorphic function F on Ω satisfying $F = f$ on V, and $\int_\Omega |F|^p \, d\mu < \infty$, where $d\sigma$ and $d\mu$ are Lebesgue measures on Ω and V, respectively.*

We set

$$h_i(x_i, \xi_i) = \phi_i(x_i) - \phi_i(\xi_i) - \phi_i'(\xi_i)(x_i - \xi_i).$$

Then we have

Lemma 1. $h_i(x_i, \xi_i) > 0$ *for* $x_i \neq \xi_i$.

Proof. We consider the case $x_i < 0$, $\xi_i \geq 0$. The other cases are similar. Then we have

$$\begin{aligned} h_i(x_i, \xi_i) &= \phi_i(-x_i) - \phi_i(\xi_i) - \phi_i'(\xi_i)(x_i - \xi_i) \\ &= -2x_i\phi_i'(\xi_i) + \tfrac{1}{2}\phi_i''(c_i)(x_i + \xi_i)^2 \quad \text{for some } c_i \neq 0. \end{aligned}$$

Since $\phi_i'(\xi_i) > 0$ for $\xi_i > 0$, we obtain the desired result.

For simplicity, we omit the index i in Lemma 2 and Lemma 3. In some neighborhood of 0, $\phi(x)$ can be written in the following form:

$$\phi(x) = b_k x^{2k} + b_{k+1} x^{2k+2} + \cdots \qquad (b_k > 0, k \geq 1).$$

Then we have the following lemma.

Lemma 2. *There exist positive constants* ε *and* c *such that*

$$h(x,\xi) \geq c[\phi''(\xi)(x-\xi)^2 + (x-\xi)^{2k}] \quad \text{for } |x| < \varepsilon, \ |\xi| < \varepsilon.$$

Proof. Let $x(t), \xi(t)$ be real analytic functions in a neighborhood of 0 satisfying $x(0) = \xi(0) = 0$. Then

$$x(t) = v_0 t^p + o(t^p), \quad \xi(t) = w_0 t^p + o(t^p),$$
$$p \geq 1, \qquad v_0^2 + w_0^2 \neq 0.$$

To prove the Lemma 2, by the curve selection lemma of Bruna-Castillo [5], it is sufficient to show that for $|t|$ sufficiently small,

$$h(x(t), \xi(t)) \geq c[\phi''(\xi(t))(x(t)-\xi(t))^2 + (x(t)-\xi(t))^{2k}].$$

Since $\phi(x) = b_k x^{2k} + b_{k+1}x^{2k+2} + \cdots$, it is easily shown that

$$h(x(t), \xi(t)) = b_k[v_0^{2k} - w_0^{2k} - 2kw_0^{2k-1}(v_0 - w_0)]t^{2pk} + o(t^{2pk}).$$

On the other hand we have

$$\begin{aligned}&\phi''(\xi(t))(x(t)-\xi(t))^2 + (x(t)-\xi(t))^{2k} \\ &\quad = [2k(2k-1)b_k w_0^{2k-2}(v_0-w_0)^2 + (v_0-w_0)^{2k}]t^{2kp} + o(t^{2pk}).\end{aligned}$$

By Diederich-Fornaess-Wiegerinck [7], there exists a positive constant δ such that

$$\begin{aligned}&v_0^{2k} - w_0^{2k} - 2kw_0^{2k-1}(v_0-w_0) \\ &\quad > \delta[w_0^{2k-2}(v_0-w_0)^2 + (v_0-w_0)^{2k}] \quad \text{for } v_0 \neq w_0.\end{aligned}$$

Therefore, if $v_0 \neq w_0$, we have for $|t|$ sufficiently small,

$$h(x(t), \xi(t)) \geq c[\phi''(\xi(t))(x(t)-\xi(t))^2 + (x(t)-\xi(t))^{2k}].$$

In case $v_0 = w_0$, we take ξ as a new parameter. Then we have

$$x(\xi) = \xi + \lambda\xi^\alpha + o(|\xi|^\alpha),$$

where α is a rational number greater than 1. We write $x(\xi)$ in the following form $x(\xi) = \xi + \xi^\alpha w(\xi)$. Then we have

$$\begin{aligned}h(x(\xi), \xi) &= \sum_{j=k}^{\infty} b_j\{(\xi + \xi^\alpha w(\xi))^{2j} - \xi^{2j} - 2j\xi^{2j-1}\xi^\alpha w(\xi)\} \\ &= \sum_{j=k}^{\infty} b_j \sum_{i=2}^{2j} \binom{2j}{i} \xi^{2j-i+\alpha i} w(\xi)^i \\ &= b_k \binom{2k}{2} \lambda^2 \xi^{2k-2+2\alpha} + o(|\xi|^{2k-2+2\alpha}).\end{aligned}$$

On the other hand,

$$\begin{aligned}&\phi''(\xi)(x(\xi)-\xi)^2 + (x(\xi)-\xi)^{2k} \\ &\quad = 2k(2k-1)b_k\lambda^2\xi^{2k-2+2\alpha} + o(|\xi|^{2k-2+2\alpha}) + \lambda\xi^{2k\alpha} + o(|\xi|^{2k\alpha}) \\ &\quad = O(|\xi|^{2k-2+2\alpha}).\end{aligned}$$

Therefore we have

$$h(x(\xi),\xi) \geq c[\phi''(\xi)(x(\xi)-\xi)^2 + (x(\xi)-\xi)^{2k}],$$

for $|\xi|$ sufficiently small. This completes the proof of Lemma 2.

Lemma 3. *Let* $\phi(x), h(x,\xi)$ *be as in Lemma 2. Then we have*

$$h(x,\xi) \geq c[\phi''(\xi)(x-\xi)^2 + (x-\xi)^{2k}]$$

for $|x| \leq a$, $|\xi| \leq a$.

Proof. From Lemma 2, there exists an $\varepsilon > 0$ such that

$$h(x,\xi) \geq c[\phi''(\xi)(x-\xi)^2 + (x-\xi)^{2k}] \quad \text{for } |x| \leq \varepsilon,\ |\xi| \leq \varepsilon.$$

If $|\xi| \geq \varepsilon$, $|x-\xi|$ small, then $|x| \geq \varepsilon/2$. Suppose that $x < 0$, $\xi > 0$. Then we obtain

$$h(x,\xi) > -2x\phi'(\xi) > c.$$

The other cases are proved similarly. This completes the proof of Lemma 3.

Thus we have proved that

$$\begin{aligned}(1)\qquad & \phi_i(x_i) - \phi_i(\xi_i) - \phi_i'(\xi_i)(x_i-\xi_i) \\ & \qquad \geq c[\phi_i''(\xi_i)(x_i-\xi_i)^2 + (x_i-\xi_i)^{2n_i}],\\ (2)\qquad & \psi_i(y_i) - \psi_i(\eta_i) - \psi_i'(\eta_i)(y_i-\eta_i) \\ & \qquad \geq c[\psi_i''(\eta_i)(y_i-\eta_i)^2 + (y_i-\eta_i)^{2m_i}]\end{aligned}$$

for $\xi_i \in [0,a]$, $\eta_i \in [0,a]$, $i = 1,\dots,N$. Without loss of generality, we may assume that $m_i \leq n_i$ for $i = 1,\dots,N$. We set, for $\zeta_j = \xi_j + i\eta_j$, $\rho_j(\zeta_j) = \phi_j(\xi_j) + \psi_j(\eta_j)$ and

$$\begin{aligned}F_j(\zeta_j, z_j) = & -2\frac{\partial\rho}{\partial\zeta_j}(\zeta)(z_j-\zeta_j) \\ & + \gamma[(\psi_j''(\eta_j) - \phi_j''(\xi_j))(z_j-\zeta_j)^2 + (z_j-\zeta_j)^{2m_j}],\end{aligned}$$

where γ will be determined later. Using (1), (2), we obtain, for $z_j = x_j + iy_j$,

$$\begin{aligned}\operatorname{Re}&(-\rho_j(\zeta_j) + \rho_j(z_j) + F_j(\zeta_j, z_j)) \\ &= -\phi_j(\xi_j) - \psi_j(\eta_j) + \phi_j(x_j) + \psi_j(y_j) \\ &\quad - [\phi_j'(\xi_j)(x_j-\xi_j) + \psi_j'(\eta_j)(y_j-\eta_j)] \\ &\quad + \gamma(\psi_j''(\eta_j) - \phi_j''(\xi_j))\{(x_j-\xi_j)^2 - (y_j-\eta_j)^2\} + \gamma\operatorname{Re}[(z_j-\zeta_j)^{2m_j}] \\ &\geq \phi_j''(\xi_j)\{(c-\gamma)(x_j-\xi_j)^2 + \gamma(y_j-\eta_j)^2\} + \gamma\operatorname{Re}[(z_j-\zeta_j)^{2m_j}] \\ &\quad + \psi_j''(\eta_j)\{(c-\gamma)(y_j-\eta_j)^2 + \gamma(x_j-\xi_j)^2\} + c(y_j-\eta_j)^{2m_j}.\end{aligned}$$

If we choose $\gamma > 0$ small enough, then it holds that (see [7, Proposition 3.5]),

$$c(y_j-\eta_j)^{2m_j} + \gamma\operatorname{Re}[(z_j-\zeta_j)^{2m_j}] \geq c|z_j-\zeta_j|^{2m_j}.$$

Therefore we have for sufficiently small γ,

$$\begin{aligned}(3)\qquad & -\rho_j(\zeta_j)+\rho_j(z_j)+\operatorname{Re}F_j(\zeta_j,z_j)\\ &\geq c[(\phi_j''(\xi_j)+\psi_j''(\eta_j))|z_j-\zeta_j|^2+|z_j-\zeta_j|^{2m_j}].\end{aligned}$$

Now we fix γ so that the inequality (3) holds. We set

$$F(\zeta,z)=\sum_{j=1}^{N}F_j(\zeta_j,z_j).$$

Then we obtain

$$\begin{aligned}(4)\qquad & -\rho(\zeta)+\rho(z)+\operatorname{Re}F(\zeta,z)\\ &\geq c\sum_{j=1}^{N}\{(\phi_j''(\xi_j)+\psi_j''(\eta_j))|z_j-\zeta_j|^2+|z_j-\zeta_j|^{2m_j}\}\end{aligned}$$

for $(\zeta,z)\in\Omega\times\Omega$. We set, for $\zeta_j=\xi_j+i\eta_j\in\Omega$, $z_j=x_j+iy_j\in\Omega$, $j=1,\dots,N$,

$$P_j(\zeta_j,z_j)=-2\frac{\partial\rho_j}{\partial\zeta_j}(\zeta_j)+\gamma(\psi_j''(\eta_j)-\phi_j''(\xi_j))(z_j-\zeta_j)+(z_j-\zeta_j)^{2m_j-1}.$$

Then

$$F(\zeta,z)=\sum_{j=1}^{N}P_j(\zeta_j,z_j)(z_j-\zeta_j).$$

For $\varepsilon>0$ sufficiently small, we set

$$V_\varepsilon=\{z\in V:\rho(z)<-\varepsilon\}\quad\text{and}\quad\Omega_\varepsilon=\{z\in\Omega:\rho(z)<-\varepsilon\}.$$

Let f^* be the boundary value of $f\in H^p(V)$ $(1\leq p<\infty)$. Then $f^*\in L^p(\partial V)$. Now we are going to prove the following.

Proposition 1. *For $f\in H^p(V)$ $(1\leq p<\infty)$, and $z\in V$, we have the formula*

$$f(z)=\int_{\partial V}f^*(\zeta)K(\zeta,z),$$

where

(i) *$K(\zeta,z)$ is written as*

$$\sum_{\substack{K=(k_1,\dots,k_{N-m-1})\\ S=(\alpha_1,\dots,\alpha_{N-m})}}\frac{\alpha_{K,S}(\zeta,z)\bigwedge_{j=1}^{N-m-1}\overline{\partial}_\zeta P_{k_j}\bigwedge_{i=1}^{N-m}d\zeta_{\alpha_i}}{F(\zeta,z)^{N-m}}$$

where $k_1,\dots,k_{N-m-1}$ are positive integers between 1 *and N which are different from each other.*

(ii) *$\alpha_{K,S}(\zeta,z)$ are smooth on $\overline{\Omega}\times\overline{\Omega}$, holomorphic in $z\in\Omega$.*

Proof. We follow the proof of Lemma I.1 of Stout [13]. Fix $z\in V$. As $f\in H^p(V)$, f is holomorphic in a neighborhood of $\overline{V}_\varepsilon$ in V. By the assumption imposed on V, there exists $\varepsilon_1 > 0$ such that (i) $z \in V_{\varepsilon_1}$ (ii) $\partial h_1 \wedge \cdots \wedge \partial h_m \wedge \partial\rho \neq 0$ on ∂V for $\varepsilon \in [0, \varepsilon_1]$. So by the theorem of Hatziafratis [8], we have

$$f(z) = \int_{\partial V_\varepsilon} f(w) K(w, z) \quad \text{for } \varepsilon \in (0, \varepsilon_1].$$

If we choose ε_1 sufficiently small, there exist a continuous function $\Delta\colon \partial V \times [0, \varepsilon_1] \to C$, and a smooth map $\chi\colon \partial V \times [0, \varepsilon_1] \to \overline{V}$ such that for fixed ε, $\chi(\cdot, \varepsilon)$ takes ∂V diffeomorphically onto the surface ∂V_ε, $\chi(\cdot, 0)$ the identity, and such that

$$\int_{\partial V_\varepsilon} f(w) K(w, z) = \int_{\partial V} f(\chi(w, \varepsilon)) K(\chi(w, \varepsilon), z) \Delta(w, \varepsilon).$$

Since $\chi(w, \varepsilon) \to w$, nontangentially as $\varepsilon \to 0+$, the dominated convergence theorem implies (cf. Stein [11]) that the integral on the right tends, as $\varepsilon \to 0+$, to $\int_{\partial V} f^*(w) K(w, z)$. This completes the proof of Proposition 1.

We set

$$F(z) = \int_{\partial V} f^*(\zeta) K(\zeta, z) \quad \text{for } z \in \Omega.$$

Then $F(z)$ is a holomorphic function in Ω which satisfies $F(z) = f(z)$ for $z \in V$. Let $\phi(x)$ be the function as in Lemma 2. Then we have

Lemma 4. *Let A be a positive number, close to* 0, *and q be a positive integer. Let $x \in [-R, R]$, $0 < R < a/2$. We set*

$$I_1 = \iint_{|\tau|<R; |\sigma|<R} \frac{|\phi'''(x - \sigma)| |\tau| \, d\sigma \, d\tau}{[A + \phi''(x - \sigma)(\tau^2 + \sigma^2) + (\tau^2 + \sigma^2)^k]^q},$$

$$I_2 = \iint_{|\tau|<R; |\sigma|<R} \frac{|\phi'''(x - \tau)| |\tau| \, d\tau \, d\sigma}{[A + \phi''(x - \tau)(\tau^2 + \sigma^2)]^q},$$

$$I_3 = \iint_{|\tau|<R; |\sigma|<R} \frac{\phi''(x - \tau) \, d\tau \, d\sigma}{[A + \phi''(x - \tau)(\tau^2 + \sigma^2)]^q}.$$

Then, for $j = 1, 2, 3$,

$$|I_j| \leq c |\log A| \quad \textit{if } q = 1,$$

$$|I_j| \leq c A^{1-q} \quad \textit{if } q > 1,$$

where c is independent of x.

Proof. There exists $\varepsilon > 0$ such that, for $|x| < \varepsilon$,

$$\phi(x) = b_k x^{2k} + b_{k+1} x^{2k+2} + \cdots \qquad (b_k > 0, k \geq 1).$$

We may assume that $|x - \sigma| \leq \varepsilon' < \varepsilon$. Then

$$\phi''(x - \sigma) \geq c(x - \sigma)^{2k-2}, \quad |\phi'''(x - \sigma)| \leq c|x - \sigma|^{2k - \mu(k)},$$

where $\mu(k) = 3$ when $k \geq 2$,

$$\mu(k) = 1, -1, -3, \ldots, \quad \text{when } k = 1.$$

Therefore we have

$$I_1 \le c \iint_{|\tau|<R\,;|\sigma|<R} \frac{|x-\sigma|^{2k-\mu(k)}|\tau|\,d\tau\,d\sigma}{[A+|x-\sigma|^{2k-2}(\tau^2+\sigma^2)+(\tau^2+\sigma^2)^k]^q},$$

$$I_2 \le c \iint_{|\tau|<R\,;|\sigma|<R} \frac{|x-\tau|^{2k-\mu(k)}|\tau|\,d\tau\,d\sigma}{[A+|x-\tau|^{2k-2}(\tau^2+\sigma^2)]^q},$$

$$I_3 \le c \iint_{|\tau|<R\,;|\sigma|<R} \frac{|x-\tau|^{2k-2}\,d\tau\,d\sigma}{[A+|x-\tau|^{2k-2}(\tau^2+\sigma^2)]^q}.$$

From the Lemma 4.1 of Diederich-Fornaess-Wiegerinck [7], we obtain the desired results. This completes the proof of Lemma 4.

Let B_i $(i=0,1,\dots,M)$, be balls with centers on ∂V and radius r_0 which form a cover of ∂V. Let $\widetilde{B}_i$ be the ball with the same center as B_i and radius $2r_0$. We set $N-m=s$. Since $\partial h_1\wedge\cdots\wedge\partial h_m\wedge\partial\rho\ne 0$ on ∂V, we may assume, for r_0 sufficiently small,

(i) $$\left|\frac{\partial\rho}{\partial\overline{z}_s}(z)\right| \ge c > 0 \quad \text{for } z\in\widetilde{B}_0,$$

(ii) $$V\cap\widetilde{B}_0 = \{z\in\widetilde{B}_0 : z_{s+1}=\cdots=z_N=0\}.$$

Then

$$L_j = \frac{\partial\rho}{\partial\overline{z}_s}(z)\frac{\partial}{\partial\overline{z}_j} - \frac{\partial\rho}{\partial\overline{z}_j}(z)\frac{\partial}{\partial\overline{z}_s} \qquad (j=1,\dots,s-1)$$

form a base for the $(0,1)$ tangential vector fields on $\partial V\cap\widetilde{B}_0$. By a simple computation, we have

$$|L_jP_i| \le \delta_{ji}c[\phi_i''(\xi_i)+\psi_i''(\eta_i)+\gamma(|\psi_i'''(\eta_i)|+|\phi_i'''(\xi_i)|)|z_i-\zeta_i|],$$

for $i\ne s$,

$$|L_jP_s| \le c\left|\frac{\partial\rho}{\partial\overline{z}_j}\right| \le c[|\phi_j'(\xi_j)|+|\psi_j'(\eta_j)|].$$

3. Proof of Theorem 1

Without loss of generality, it is sufficient to show that

(5) $$\sup_{\varepsilon>0}\int_{\partial\Omega_\varepsilon\cap B_0} |F(z)|^p\,dS_\varepsilon(z) < \infty,$$

where $dS_\varepsilon(z)$ is the element of surface area on $\partial\Omega_\varepsilon$. We set

$$\tau_j=\operatorname{Re}(z_j-\zeta_j),\quad \sigma_j=\operatorname{Im}(z_j-\zeta_j),\qquad j=1,\dots,N,$$
$$\lambda=\operatorname{Im}F(\zeta,z),\quad \rho=\rho(\zeta)-\rho(z).$$

By the transversality of V, for $\zeta\in\partial\Omega\cap\widetilde{B}_0$ fixed, τ_j, σ_j, λ, ρ $(j=1,\dots,s-1,s+1,\dots,N)$ form coordinates of $\overline{\Omega}\cap\widetilde{B}_0$ in such a way that τ_j, σ_j, λ

$(j = 1, \dots, s-1)$ form coordinates of $\partial V \cap \widetilde{B}_0$ for $z \in \overline{\Omega} \cap \widetilde{B}_0$ fixed. We set

$$H(z) = \int_{\partial V \cap \widetilde{B}_0} f^*(\zeta) K(\zeta, z).$$

Then we have

$$\begin{aligned}
&\int_{\partial \Omega_\varepsilon \cap B_0} |H(z)|\, dS_\varepsilon(z) \\
&\quad \le \int_{\partial V \cap \widetilde{B}_0} |f^*(\zeta)| \left(\int_{\partial \Omega_\varepsilon \cap B_0} |K(\zeta, z)|\, dS_\varepsilon(z) \right) d\sigma(\zeta) \\
&\quad \le c \int_{\partial V \cap \widetilde{B}_0} |f(\zeta)|\, d\sigma(\zeta) \\
&\quad\quad \times \int_{\partial \Omega_\varepsilon \cap B_0} \frac{\prod_{j=1}^{s-1} \{ \phi_j''(\xi_j) + \psi_j''(\eta_j) + (|\psi_j'''(\eta_j)| + |\phi_j'''(\xi_j)|)|z_j - \zeta_j| \}\, dS_\varepsilon(z)}{[|\rho(z)| + \sum_{j=1}^N \{(\phi_j''(\xi_j) + \psi_j''(\eta_j))|\zeta_j - z_j|^2 + |\zeta_j - z_j|^{2m_j}\} + |\lambda|]^s}.
\end{aligned}$$

Using Lemma 4, we obtain

$$\begin{aligned}
&\int_{\partial \Omega_\varepsilon \cap B_0} |H(z)|\, dS_\varepsilon(z) \\
&\quad \le c \int_{\partial V \cap \widetilde{B}_0} |f^*(\zeta)|\, d\sigma(\zeta) \int_{\substack{|\tau_s| < R \\ \cdots \\ |\sigma_n| < R}} \left| \log \left(\varepsilon + \sum_{j=s+1}^N (\tau_j^{2m_j} + \eta_j^{2m_j}) \right) \right| d\tau_s \cdots d\sigma_n \\
&\quad \le c \int_{\partial V} |f^*(\zeta)|\, d\sigma(\zeta).
\end{aligned}$$

Thus we have proved the inequality (5) when $p = 1$. In case $p > 1$, we take q such that $1/p + 1/q = 1$. Let $z \in \widetilde{B}_0$. Then for $\zeta \in \partial V \cap \widetilde{B}_0$, we can write

$$K(\zeta, z) = T(\zeta, z)\, d\tau_1 \wedge d\sigma_1 \wedge \cdots \wedge d\tau_{s-1} \wedge d\sigma_{s-1} \wedge d\lambda.$$

By applying Hölder's inequality, we have

$$|H(z)|^p \le c \left(\int_{\partial V \cap \widetilde{B}_0} |f(\zeta)|^p |T(\zeta, z)|\, d\sigma(\zeta) \right) \left(\int_{\partial V \cap \widetilde{B}_0} |T(\zeta, z)|\, d\sigma(\zeta) \right)^{p/q}.$$

By applying the method in case $p = 1$, we can easily prove the inequality (5), which completes the proof of Theorem 1.

4. Proof of Theorem 2

We set $B(\zeta, z) = -2\rho(\zeta) + F(\zeta, z)$ and $B_\varepsilon(\zeta, z) = -2\rho(\zeta) + F(\zeta, z) - 2\varepsilon$. Then we obtain

$$\begin{aligned}
\operatorname{Re} B(\zeta, z) &\ge -\rho(\zeta) - \rho(z) \\
&\quad + c \sum_{j=1}^N \{ (\phi_j''(\xi_j) + \psi_j''(\eta_j)) |z_j - \zeta_j|^2 + |z_j - \zeta_j|^{2m_j} \},
\end{aligned}$$

for $(\zeta, z) \in \overline{\Omega} \times \overline{\Omega}$. The integral formula in Proposition 1 also holds when we replace $F(\zeta, z)$ by $B_\varepsilon(\zeta, z)$, V by V_ε, and f^* by f. Since V is nonsingular, by applying Stokes' theorem, we have letting $\varepsilon \to 0$,

$$f(z) = \int_V f(\zeta) \overline{\partial}_\zeta K(\zeta, z) \quad \text{for } z \in V.$$

Each term of $\bar{\partial}_\zeta K(\zeta, z)$ consists of

$$\frac{\bar{\partial}_\zeta \alpha_{K,S}(\zeta, z) \bigwedge_{j=1}^{s-1} \bar{\partial}_\zeta P_{k_j} \bigwedge_{i=1}^{s} d\zeta_{\alpha_i}}{B(\zeta, z)^s} \quad \text{and} \quad \frac{\alpha_{K,S}(\zeta, z) \bigwedge_{j=1}^{s} \bar{\partial}_\zeta P_{k_j} \bigwedge_{i=1}^{s} d\zeta_{\alpha_i}}{B(\zeta, z)^{s+1}},$$

where $\alpha_{K,S}$ are smooth functions. We set

$$F(z) = \int_V f(\zeta) \bar{\partial}_\zeta K(\zeta, z) \quad \text{for } z \in \Omega.$$

Then $F(z)$ is the holomorphic function in Ω which satisfies $F(z) = f(z)$ for $z \in V$. Using the same method as in the proof of Theorem 1, we can obtain the desired result. This completes the proof of Theorem 2.

REFERENCES

1. K. Adachi, *Extending bounded holomorphic functions from certain subvarieties of a strongly pseudoconvex domain*, Bull. Fac. Sci. Ibaraki Univ. Ser. A No. 8 (1976), 1–7.
2. ——, *Extending bounded holomorphic functions from certain subvarieties of a weakly pseudoconvex domain*, Pacific J. Math. **110** (1984), 9–19.
3. ——, *Continuation of bounded holomorphic functions from certain subvarieties to weakly pseudoconvex domains*, Pacific J. Math. **130** (1987), 1–8.
4. F. Beatrous, *L^p estimates for extensions of holomorphic functions*, Michigan Math. J. **32** (1985), 361–380.
5. J. Bruna and J. del Castillo, *Hölder and L^p estimates for the ∂-equation in some convex domains with real anaytic boundary*, Math. Ann. **269** (1984), 527–539.
6. A. Cumenge, *Extension dans des classes de Hardy de fonctions holomorphes et estimation de type "mesures de Carleson" pour l'equation ∂*, Ann. Inst. Fourier **33** (1983), 59–97.
7. K. Diederich, J. E. Fornaess and J. Wiegerinck, *Sharp Hölder estimates for $\bar{\partial}$ on ellipsoids*, Manuscripta Math. **56** (1986), 399–417.
8. T. E. Hatziafratis, *Integral representation formulas on analytic varieties*, Pacific J. Math. **123** (1986), 71–91.
9. ——, *An explicit extension formula of bounded holomorphic functions from analytic varieties to strictly convex domains*, J. Funct. Anal. **70** (1987), 289–303.
10. G. M. Henkin, *Continuation of bounded holomorphic functions from submanifolds in general position to strictly pseudoconvex domains*, Math. USSR-Izv. **6** (1972), 536–563.
11. E. M. Stein, *Boundary behavior of holomorphic functions of several complex variables*, Princeton Univ. Press, Princeton, N.J., 1972.
12. E. L. Stout, *an integral formula for holomorphic functions on strictly pseudoconvex hypersurfaces*, Duke Math. J. **42** (1975), 347–356.
13. ——, *H^p-functions on strictly pseudo-convex domains*, Amer. J. Math. **98** (1976), 821–852.

DEPARTMENT OF MATHEMATICS, NAGASAKI UNIVERSITY, NAGASAKI 852, JAPAN